

Quasi-Coulomb series in a two-dimensional three-body system

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We show that the bound states in a three-body system display a Coulomb series with a Gaussian cut-off provided: (i) the system consists of a light particle and two heavy bosonic ones, (ii) the heavy-light short-range potential has a resonance in the p -wave scattering amplitude, and (iii) all three particles move in two space dimensions. For a decreasing mass ratio this quasi-Coulomb series merges into a pure Coulombic one.

Introduction.— Many physical phenomena depend crucially on the number of dimensions of space and two dimensions play a very special role. Three examples may illustrate this point. In two dimensions (i) the Huygens principle familiar from classical optics fails [1], (ii) in quantum mechanics already an infinitesimally small binding potential suffices to create a bound state [2], and (iii) the Berezinskii-Kosterlitz-Thouless transition [3] takes place only there. In the realm of few-body physics, the Efimov effect [4], which is the formation of an infinite series of three-body bound states induced by a two-body s -wave resonance, only exists in three dimensions [5]. However, in the present Letter we show that even in two dimensions there exists an infinite series of three-body bound states provided the two-body interaction has a p -wave resonance.

The bound states of three particles associated with the Efimov effect result from an effective potential, which decays with the *square* of the distance [4, 6–8] between the particles. In the case of a p -wave resonance, an universal series of three-body bound states is formed due to an effective potential, which decays [9, 10] with the *cube* of the distance. Consequently, in the Efimov effect the energy spectrum depends exponentially [4, 11, 12] on the quantum number n , whereas in the case of the p -wave resonance it scales with the sixth power [9].

We emphasize that the shape of the effective potentials and the associated energy spectra are intimately connected to the fact that the particles experience a three-dimensional space. In this Letter we confine a three-body system composed of a light atom of mass m and two heavy bosonic ones of mass M with a p -wave resonance in the heavy-light interaction to two space dimensions [13, 14]. We derive an effective potential, which consists of the familiar Efimov potential screened by the inverse of a logarithm. The length scale of the logarithm is determined by the parameters of the two-dimensional p -wave resonance.

The appearance of the logarithm is a consequence of the dimensionality of the problem. Indeed, it originates from the asymptotic behavior of the two-dimensional

scattering amplitude at low incident energy [15, 16] given by a Hankel function, which displays a logarithmic singularity at the origin [17].

The Langer transformation [18] applied to the two-dimensional radial Schrödinger equation containing the product of the Efimov potential and the inverse of the logarithm gives rise to an effective one-dimensional Coulomb potential [19] with the familiar discrete Coulomb energy spectrum. Due to this transformation the energy eigenvalue is multiplied by an exponential and we arrive at the universal energy spectrum

$$E_n = -\frac{E_0}{n^2} \exp\left[-\frac{\pi^2}{2} \frac{\mu}{M} n^2\right] \quad (1)$$

for large n . The familiar Coulomb series is modified by a Gaussian cut-off governed by the mass ratio μ/M , where $\mu \equiv 2mM/(m+2M)$ denotes the reduced mass and E_0 is the characteristic energy determined by the short-range physics.

Similar to the Efimov effect [8], we can detect these three-body bound states using an atomic mixture with a large mass ratio. Indeed, mixtures with a p -wave resonance have already been realized with K and Rb [20], as well as with Li and Rb [21], corresponding to the mass ratios $m_K/M_{Rb} \approx 0.5$ and $m_{Li}/M_{Rb} \approx 0.1$.

Binding energy for light particle.— Since our three-body system consists of a light particle which interacts with two heavy particles, we employ [9, 10, 22, 23] the Born-Oppenheimer approximation [15]. As in the case of three dimensions [9], the energy spectrum of this system is determined by the relative motion of the two heavy particles dictated by the Schrödinger equation

$$\left\{ \Delta_{\mathbf{R}}^{(2)} + \frac{M}{\hbar^2} [E - \mathcal{V}(\mathbf{R})] \right\} \chi(\mathbf{R}) = 0. \quad (2)$$

Here \mathbf{R} and E denote the separation between the two heavy particles and the total three-body energy, respectively, and $\Delta_{\mathbf{R}}^{(2)}$ is the Laplacian in two dimensions.

The effective interaction potential

$$\mathcal{V}(\mathbf{R}) \equiv -\frac{[\hbar\kappa(\mathbf{R})]^2}{2\mu} \quad (3)$$

is the bound-state energy of the light particle interacting with two non-moving heavy particles [25].

The values of $\kappa = \kappa(R)$ follow [9, 26] from the condition that the determinant of the system of linear algebraic equations

$$C_m^{(+)} + i\pi T_m(i\kappa) \sum_{m'=-\infty}^{\infty} H_{m-m'}^{(1)}(i\kappa R) C_{m'}^{(-)} = 0 \quad (4)$$

$$C_m^{(-)} + i\pi T_m(i\kappa) \sum_{m'=-\infty}^{\infty} H_{m'-m}^{(1)}(i\kappa R) C_{m'}^{(+)} = 0 \quad (5)$$

for the coefficients $C_m^{(\pm)}$ vanishes, which provides us with a transcendental equation. Here $m = 0, \pm 1, \pm 2, \dots$ and $H_m^{(1)}$ are the Hankel functions [17] with the on-shell T -matrix elements [27]

$$T_m(i\kappa) = -\frac{1}{\pi} \frac{1}{\cot[\delta_m(i\kappa)] - i} \quad (6)$$

of the heavy-light potential U determined by the scattering phase δ_m . For the sake of simplicity U is assumed to be spherically symmetric and to have the finite range r_0 , that is $U(r > r_0) = 0$.

We emphasize that a similar set of equations emerges [9] in the three-dimensional case. However, the main difference in two dimensions is the appearance of the Hankel functions $H_m^{(1)}$ with an integer rather than a half-integer index. This substitution is a consequence of the reduced dimensionality and can be traced back [26] to the Green function [15] of a free particle in two dimensions given by $H_0^{(1)}$.

Efficiency of s - and p -wave scattering.— Since there is no Efimov effect in two dimensions [5, 26, 28], we now focus on the p -wave resonant state in the heavy-light potential U and show that the s - and p -wave scattering are of the same order. This feature is crucial for truncating the system Eqs. (4)-(5).

Similar to the three-dimensional scattering, in the low-energy limit, that is for $\kappa r_0 \ll 1$, the p -wave scattering phase

$$\cot[\delta_1(i\kappa)] \cong \frac{2}{\pi} \left[\frac{1}{a_1 \kappa^2} + \ln(i\kappa r_1) \right] \quad (7)$$

is parametrized within the two-dimensional effective-range expansion [29] by the p -wave effective scattering length a_1 and effective range r_1 . Both a_1 and r_1 are non-negative parameters and for $a_1 \gg r_1^2$ they are determined by the energy

$$\varepsilon_1 \cong -\frac{\hbar^2}{\mu a_1 \ln[a_1/(2r_1^2)]} \quad (8)$$

of the p -wave bound state. The latter is defined by the positive-valued pole κ_1 of the matrix element T_1 , given by Eq. (6) with Eq. (7), that is $\varepsilon_1 \equiv -(\hbar\kappa_1)^2/2\mu$.

Moreover, in the case of a p -wave resonance, that is $a_1^{-1} = 0$, the effective range r_1 has an upper bound [29] determined by the potential range r_0 , that is $r_1 \leq \frac{1}{2}e^\gamma r_0$, with γ being the Euler constant, for any two-dimensional short-range potential. The matrix element T_1 for the resonant channel given by Eqs. (6) and (7) is of the same order as T_0 for the non-resonant s -wave channel [15] following from

$$\cot[\delta_0(i\kappa)] \cong \frac{2}{\pi} \left[\gamma + \ln\left(\frac{i\kappa a_0}{2}\right) \right]. \quad (9)$$

Here a_0 is the two-dimensional scattering length.

Effective potential for p -wave resonance.— Since in the low-energy limit, that is $\kappa r_0 \ll 1$, we have the estimate $T_{m>1}(i\kappa) \sim (\kappa r_0)^{2m}$, we can confine ourselves in Eqs. (4) and (5) only to the s - and p -waves, giving rise to a system of six algebraic equations for $C_0^{(\pm)}$, $C_1^{(\pm)}$, and $C_{-1}^{(\pm)}$. This system has non-trivial solutions only if the corresponding determinant vanishes, which provides us with two branches of solutions [26] for κ corresponding to different wave functions of the light particle.

The first branch is determined [26] by the transcendental equation

$$K_0\left(\xi \frac{R}{r_1}\right) - K_2\left(\xi \frac{R}{r_1}\right) = \pm \left(\frac{r_1^2}{a_1} \frac{1}{\xi^2} + \ln \xi \right) \quad (10)$$

for $\xi \equiv \kappa r_1$, where K_m are modified Bessel functions [17].

In the resonant case, that is $a_1^{-1} = 0$, Eq. (10) has a solution ξ_1 only for the plus sign on the right-hand side due to the asymptotic behavior [17] $K_2(z) \cong 2/z^2$ and $K_0(z) \cong -\ln(z)$ for $z \rightarrow 0$. In the limit of $0 < \xi_1 \ll 1$, we find $\xi_1^2 \cong 2e^{1-2\gamma}[\rho^2 \ln(\rho \ln \rho)]^{-1}$ for $\rho \equiv \exp(\frac{1}{2} - \gamma) R r_1^{-1} \gg 1$, giving rise to the effective potential

$$\mathcal{V}_1^{(0)}(R) \cong -\frac{\hbar^2}{\mu R^2} \frac{1}{\ln \frac{R}{r_1} - \gamma + \frac{1}{2} + \ln\left(\ln \frac{R}{r_1} - \gamma + \frac{1}{2}\right)} \quad (11)$$

for $R \gg r_1$. Here the subscript I refers to the first branch.

However, near the resonance, $a_1 \gg r_1^2$, that is in the case of the weakly-bound p -wave state in U , the form of the effective potentials

$$\mathcal{V}_I^{(\pm)} \equiv -\frac{\hbar^2}{2\mu r_1^2} \left(\xi_I^{(\pm)} \right)^2$$

is determined by the two solutions $\xi_I^{(\pm)}$ of Eq. (10). The ranges of $\mathcal{V}_I^{(+)}$ and $\mathcal{V}_I^{(-)}$ are identical and equal to

$$R_1 \equiv \frac{\hbar}{\sqrt{2\mu|\varepsilon_1|}} \cong \left(\frac{a_1}{2} \ln \frac{a_1}{2r_1^2} \right)^{\frac{1}{2}}. \quad (12)$$

For large distances, $R \gtrsim R_1$, they approach exponentially the bound state energy ε_1 of the light particle defined by Eq. (8). For short distances, $R \lesssim R_1$, $\mathcal{V}_I^{(+)}$

decreases monotonically and approaches $\mathcal{V}_I^{(0)}$, Eq. (11), $\mathcal{V}_I^{(+)}(R \lesssim R_1) \cong \mathcal{V}_I^{(0)}(R)$, whereas $\mathcal{V}_I^{(-)}$ increases monotonically and vanishes at $R = \sqrt{2a_1}$.

The second branch of solutions of the truncated system of equations, Eqs. (4)-(5), is determined [26] by the transcendental equation

$$\left[K_2 \left(\xi \frac{R}{r_1} \right) + K_0 \left(\xi \frac{R}{r_1} \right) \pm \left(\frac{r_1^2}{a_1} \frac{1}{\xi^2} + \ln \xi \right) \right] \times \\ \left[K_0 \left(\xi \frac{R}{r_1} \right) \mp \ln \left(\xi \frac{e^\gamma a_0}{2 r_1} \right) \right] = 2K_1^2 \left(\xi \frac{R}{r_1} \right) \quad (13)$$

for $\xi \equiv \kappa r_1$.

In the resonant case, that is $a_1^{-1} = 0$, Eq. (13) has a solution ξ_2 only for the plus sign on the left-hand side and, in the limit of $0 < \xi_2 \ll 1$, we find $\xi_2^2 \cong 2e^{1-2\gamma} [\rho^2 (\ln \rho + 2\gamma + 1 - \ln 2)]^{-1}$ for $\rho \gg 1$, giving rise to the effective potential

$$\mathcal{V}_{II}^{(0)}(R) \cong -\frac{\hbar^2}{\mu R^2} \frac{1}{\ln \frac{R}{2r_1} + \gamma + \frac{3}{2}} \quad (14)$$

for $R \gg r_1$. Here the subscript II refers to the second branch. For a more detailed discussion of the potentials $\mathcal{V}_{II}^{(\pm)}$ arising in the second branch, we refer to Ref. [26].

On the p -wave resonance and for $R \rightarrow \infty$, both potentials $\mathcal{V}_I^{(0)}$ and $\mathcal{V}_{II}^{(0)}$ given by Eqs. (11) and (14) approach the same behavior

$$\mathcal{V}_I^{(0)}(R \rightarrow \infty) \cong \mathcal{V}_{II}^{(0)}(R \rightarrow \infty) \cong \mathcal{V}(R) \equiv -\frac{\hbar^2}{\mu R^2} \frac{1}{\ln \frac{R}{r_1}} \quad (15)$$

determined by effective range r_1 of the p -wave.

Energy spectrum induced by effective potential.—Next we focus on the two-dimensional dynamics of the two heavy bosonic particles dictated by the Schrödinger equation (2) with the potential \mathcal{V} given by Eq. (15). In particular, we show that \mathcal{V} supports an infinite number of three-body bound states following a quasi-Coulomb series.

For the case of zero orbital angular momentum between the two heavy bosonic particles, we can use the WKB solution [26]

$$\chi_E(R) \propto \sin[\varphi(R, R_E)] \quad (16)$$

of Eq. (2) with the phase

$$\varphi(R, R_E) \equiv \frac{1}{\hbar} \int_R^{R_E} dR' \sqrt{M[E - \mathcal{V}(R')]} + \theta_E \quad (17)$$

accumulated between R and the outer turning point R_E determined by the condition $\mathcal{V}(R_E) = E$. Here the phase θ_E , with $|\theta_E| \leq \pi$, which depends only weakly on the energy E , is determined by the behavior of the potential at short distances, that is for $R \sim r_1 \sim r_0$.

Since the spectrum of the bound states in \mathcal{V} has an accumulation point at $E = 0$, we need to know the behavior of φ as $E \rightarrow 0$. For this purpose, we neglect in

the limit of small and negative energy, $|E| \ll \hbar^2/(Mr_1^2)$, and large distances, $r_1 \ll R \ll R_E$, the energy E under the square root on the right-hand side of Eq. (17) and obtain with Eq. (15) the approximation

$$\varphi \cong \sqrt{\frac{M}{\mu}} \int_R^{R_E} \frac{dR'}{R' \sqrt{\ln \frac{R'}{r_1}}} + \theta_0$$

for the accumulated phase [26], or

$$\varphi(R, R_E) \cong 2\sqrt{\frac{M}{\mu}} \left[\left(\ln \frac{R_E}{r_1} \right)^{\frac{1}{2}} - \left(\ln \frac{R}{r_1} \right)^{\frac{1}{2}} \right] + \theta_0. \quad (18)$$

The energy spectrum of the weakly bound states induced by the potential \mathcal{V} follows from the familiar WKB quantization rule $\varphi(r_1, R_n) = \pi n$, giving rise to the discrete positions $R_n \cong r_1 \exp[(\mu/4M)\pi^2 n^2]$ of the outer turning points for $n \gg 1$. The connection $E_n = \mathcal{V}(R_n)$ between the binding energy E_n and R_n finally yields the asymptotic energy spectrum, Eq. (1), in the form of the Coulomb series with a Gaussian cut-off governed by the mass ratio. The characteristic energy $E_0 \sim \hbar^2/(\mu r_0^2)$ is determined by the short-range physics of \mathcal{V} .

As a result, an exact two-dimensional p -wave resonance in the heavy-light short-range interaction potential creates the two long-range effective potentials $\mathcal{V}_I^{(0)}$, Eq. (11), and $\mathcal{V}_{II}^{(0)}$, Eq. (14). They both merge into the same asymptotic potential \mathcal{V} , Eq. (15), which gives rise to an infinite series of the weakly bound three-body states.

However, near the resonance, that is for large but finite values of a_1 , the range R_1 given by Eq. (12) is finite and the effective potentials $\mathcal{V}_I^{(0)}$ and $\mathcal{V}_{II}^{(0)}$ are valid only in the region $r_1 \lesssim R \lesssim R_1$. As a result, the number N_0 of bound states is finite and given by the number of nodes of the zero-energy solution χ_0 defined by Eq. (16) with the phase $\varphi(R, R_1)$, Eq. (18), accumulated between R and $R_E \sim R_1$. Since $R_1 \gg r_1$, we can estimate $N_0 \cong \varphi(r_1, R_1)/\pi$ with the help of Eq. (12) as

$$N_0 \cong \frac{2}{\pi} \left(\frac{M}{\mu} \ln \frac{R_1}{r_1} \right)^{\frac{1}{2}} \cong \frac{1}{\pi} \left(2 \frac{M}{\mu} \ln \frac{a_1}{2r_1^2} \right)^{\frac{1}{2}}. \quad (19)$$

Thus, N_0 increases with the square root of M/μ and diverges weakly as a logarithm of a_1/r_1^2 when we tune a_1 closer to the p -wave resonance, that is for $a_1 \rightarrow \infty$.

New states as resonances in atom-molecule scattering.—The appearance of the binding potentials can be verified experimentally by scattering a heavy atom off the diatomic molecule consisting of the heavy and the light atom [23, 24]. The predicted three-body bound states manifest themselves as resonances in the cross section of the atom-molecule scattering when we tune the scattering length with Feshbach resonances [30] and approach in this way the two-dimensional p -wave resonance.

Indeed, at the low incident energy $(\hbar k)^2/M \equiv E - \varepsilon_1$, such as $kr_1 \ll 1$, the total atom-molecule cross-section [15]

$$\sigma_0 = \frac{\pi^2}{k} \left[\frac{\pi^2}{4} + \ln^2 \left(\frac{kA_0}{2} e^\gamma \right) \right]^{-1}$$

is determined mainly by the two-dimensional atom-molecule scattering length A_0 . For large values of a_1 , it can be estimated by matching at $R \sim R_1$ the logarithmic derivative of $\chi_0(R) \propto \sin[\varphi(r_1, R)]$ with φ for $R \lesssim R_1$, with the logarithmic derivative of the spherically-symmetric zero-energy solution $\chi(R) \propto \ln(R/A_0)$ of Eq. (2) valid for $R \gtrsim R_1$, where the effective potential \mathcal{V} vanishes. Indeed, with the help of Eq. (18) for $\varphi(r_1, R)$, we arrive at

$$\frac{1}{\ln(R_1/A_0)} = R_1 \frac{\chi'_0(R_1)}{\chi_0(R_1)} = \frac{2M \cot(\pi N_0)}{\mu \pi N_0}$$

with N_0 given by Eq. (19).

Hence, the atom-molecule scattering length

$$A_0 = \left(\frac{a_1}{2} \ln \frac{a_1}{2r_1^2} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\mu}{2M \cot[\pi N_0(a_1)]} \right\}$$

exhibits an infinite series of resonances at $a_1 = a_1^{(n)}$, where $a_1^{(n)}$ are the zeros of the equation $N_0(a_1^{(n)}) = (n + \frac{1}{2})$. For $n \gg 1$, Eq. (19) provides with the expression

$$a_1^{(n)} \sim r_1^2 \exp \left[\frac{\pi^2}{2} \frac{\mu}{M} n^2 \right]$$

for the position of the resonances and for a small mass-ratio many resonances are visible.

Summary.— We have found a novel series of bound states in a three-body system consisting of a light particle and two heavy bosonic ones when the heavy-light short-range interaction potential has a two-dimensional p -wave resonance and the system is constrained to two space dimensions. In the case of an exact resonance, the effective potentials between the two heavy particles are attractive and of long-range and support an infinite number of bound states. The spectrum has the form of the Coulomb series with a Gaussian cut-off governed by the mass ratio. We emphasize that these results are a consequence of an intricate interplay between the symmetry properties of the underlying resonances and the dimensionality of the problem. Throughout this Letter we have focused on an atomic system but we envision applications in nuclear physics as well.

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